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Generalization of Möbius-Type Arithmetic Functions and Their Algebraic Properties

Dante Rio Sebastian
Teknik dan Informatika, Indonesia
Corresponding Author: dante.dre@bsi.ac.id

Abstract. This study aims to formulate a parameterized family of Möbius-type arithmetic functions and investigate their algebraic properties, including multiplicativity, Dirichlet inverses, convolution identities, generalized Möbius inversion formulas, Dirichlet-series representations, and relationships with other arithmetic functions. The research employs a qualitative approach based on a literature review and mathematical-theoretical analysis using formal definitions, deductive proofs, prime-factorization techniques, and Dirichlet convolution theory. The results demonstrate that the proposed function, $\mu_t(n)$, preserves multiplicativity, satisfies the reduction principle to the classical Möbius function when $t = 1$, becomes the characteristic function of square-free integers when $t = -1$, and serves as the Dirichlet inverse of $z_t(n) = t^{\Omega(n)}$. Furthermore, the study derives a generalized Möbius inversion formula, establishes a new family of generalized totient functions, and obtains consistent Dirichlet-series representations. These findings provide an integrated mathematical framework for understanding generalized Möbius-type functions and extend the theoretical foundations of arithmetic identities, inversion theory, and future research in analytic number theory and the algebra of arithmetic functions.

Keywords: Arithmetic Functions; Dirichlet Convolution; Generalized Möbius Function; Möbius Inversion; Multiplicative Functions.

1. BACKGROUND

Arithmetic functions constitute one of the fundamental objects in number theory because they are used to describe various properties of positive integers based on their divisor structures. Among these functions, the Möbius function holds an important position because it connects prime factorization, square-free integers, Dirichlet convolution, and the Möbius inversion formula. Classically, the Möbius function $\mu(n)$is equal to zero when $n$contains a squared prime factor, whereas for square-free integers, its value is determined by the parity of the number of distinct prime factors. These characteristics make the Möbius function a multiplicative function that can be used to separate or invert summation relations over the divisors of an integer.

The algebraic role of the Möbius function becomes particularly evident in the structure of arithmetic functions equipped with ordinary addition and Dirichlet convolution. For two arithmetic functions $f$and g, the Dirichlet convolution is defined as follows:

Within this algebraic structure, the convolution identity function $\varepsilon$is defined by $\varepsilon(1) = 1$and $\varepsilon(n) = 0$for $n > 1$. The Möbius function is the Dirichlet inverse of the constant function $\mathbf{1}(n) = 1$, and therefore satisfies the relation $\mu * \mathbf{1} = \varepsilon$. This relation forms the basis of the Möbius inversion formula and demonstrates that the Möbius function is not merely a counting function, but also an inverse element in the algebra of arithmetic functions. Studies of convolution equations indicate that the existence of inverses and solutions to arithmetic

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equations strongly depends on the value of a function at $n = 1$and on the structure of the convolution operation being applied (Haukkanen, 2023).

Developments in number theory have shown that the classical Möbius function can be extended by introducing parameters, modifying its values at prime powers, or replacing the divisor system and convolution operation. One widely studied extension is the Möbius function of order k, which was introduced as a generalization of the classical Möbius function. Bege (2001) developed Apostol's Möbius function of order $k$by introducing an additional integer parameter, thereby producing a broader family of arithmetic functions. This generalization demonstrates that changes in parameters can influence the values of the function at prime powers and the behavior of its partial sums. Consequently, a generalization parameter does not merely produce a variation in definition, but also determines the analytical and algebraic characteristics of the resulting function.

Other generalizations have been developed by extending the Möbius inversion formula and modifying the convolution operation. He, Hsu, and Shiue (2006) constructed a broader class of inversion formulas using generalized Möbius functions and modified Dirichlet convolutions. Their results show that the Möbius inversion principle can be extended beyond its classical form, provided that the functions and operations involved satisfy certain algebraic conditions, including commutativity, the existence of an identity element, and the existence of inverses. In a different setting, Yangklan and Laohakosol (2017) defined a generalized unitary Möbius function within a commutative ring of arithmetic functions equipped with unitary convolution. This approach generated new inversion formulas and characterizations of multiplicative functions.

Generalized Möbius functions are also related to other important arithmetic functions. Laohakosol, Ruengsinsub, and Pabhapote (2006), for example, replaced the classical Möbius function in Ramanujan sums with the Souriau–Hsu–Möbius function to construct generalized Ramanujan sums. Their study demonstrated that generalizing the Möbius function can produce identities, representations, orthogonality relations, and inversion formulas that differ from their classical counterparts. Another study showed that generalized Möbius functions can be used to characterize completely multiplicative functions through particular convolution relations (Laohakosol et al., 2012). These findings confirm that changes in the definition of the Möbius function may affect its relationships with Euler's totient function, Ramanujan sums, divisor functions, and other multiplicative arithmetic functions.

Although various forms of generalized Möbius functions have been introduced, previous studies have generally examined one particular generalization or one type of convolution operation separately. Some studies have focused on Möbius functions of a particular order, while others have concentrated on inversion formulas, Ramanujan sums, or unitary convolutions. This condition indicates the need for a more integrated investigation of generalized Möbius-type arithmetic functions, particularly one that treats the generalization parameter as an element determining multiplicativity, identity, inverse, and convolution relations. An integrated analysis is necessary to identify the conditions under which a generalized function preserves the properties of the classical Möbius function and the conditions under which it produces a new algebraic structure.

The novelty of this study lies in the development of a mathematical analytical framework that connects the forms or parameters of generalized Möbius-type arithmetic functions with their resulting algebraic properties. The analysis is conducted over the domain of positive integers while maintaining prime factorization structures and the basic definition of the classical Möbius function as controlling conditions. Each generalized form is examined according to its values at prime powers, multiplicative properties, behavior under Dirichlet convolution, existence of identity and inverse elements, and relationships with other arithmetic functions. This approach is expected to provide a more systematic mapping of the algebraic consequences arising from changes in the parameters or construction rules of Möbius-type functions.

Based on the preceding discussion, this study aims to formulate generalized forms of Möbius-type arithmetic functions and analyze their associated algebraic properties. More specifically, the study seeks to determine the conditions of multiplicativity, identify identity and inverse elements under Dirichlet convolution, establish several convolution relations, and explain the relationships between generalized Möbius-type functions and other arithmetic functions. The findings are expected to broaden the understanding of the algebraic structure of arithmetic functions and provide a theoretical foundation for developing inversion formulas, divisor identities, and further generalizations of functions in number theory.

2. THEORETICAL REVIEW

Arithmetic Functions in Number Theory

An arithmetic function is a function defined on the set of positive integers and generally taking values in the real or complex numbers. Formally, an arithmetic function can be expressed as $f: \mathbb{N} \to \mathbb{C}$. Arithmetic functions are used to represent various characteristics of

integers, particularly those related to divisors, prime factorization, and divisibility. Important arithmetic functions in number theory include the identity function, divisor-counting function, sum-of-divisors function, Euler's totient function, Liouville function, and Möbius function.

The structure of arithmetic functions is examined not only through their individual values but also through algebraic operations connecting one function to another. These operations include pointwise addition, scalar multiplication, and Dirichlet convolution. Within this framework, the collection of all arithmetic functions can be regarded as a commutative algebraic structure. This perspective enables many identities in number theory, especially those involving sums over divisors, to be written more systematically and compactly (Haukkanen, 2023).

Prime factorization plays a fundamental role in the study of arithmetic functions. According to the fundamental theorem of arithmetic, every positive integer $n > 1$has a unique prime factorization of the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \qquad \text{(I)}$$

where $p_1, p_2, \ldots, p_r$are distinct prime numbers and $\alpha_i$are positive integers. Consequently, the value of a multiplicative arithmetic function at $n$can be determined from its values at the prime powers $p_i^{\alpha_i}$. This local principle at prime powers constitutes the main foundation for constructing generalizations of Möbius-type arithmetic functions.

The Classical Möbius Function

The classical Möbius function, denoted by $\mu(n)$, is one of the most important arithmetic functions in number theory. It is defined by

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = p_1 p_2 \cdots p_r, \text{ where the } p_i \text{ are distinct}, \\ 0, & p^2 \mid n \text{ for some prime } p. \end{cases} \qquad \text{(II)}$$

This definition indicates that the value of the Möbius function is determined by the prime-factorization structure of an integer. The value $\mu(n)$is zero when $n$contains a squared prime factor. When $n$is square-free with $r$distinct prime factors, its value is $(-1)^r$. Thus, the Möbius function combines two types of arithmetic information: square-freeness and the parity of the number of distinct prime factors.

The Möbius function is multiplicative. This means that for positive integers $m$and $n$satisfying $\gcd(m, n) = 1$,

$\mu(mn) = \mu(m)\mu(n)$.

This property follows from the uniqueness of prime factorization. When $\gcd(m, n) = 1$, the prime factors of $m$and $n$are disjoint. If either number contains a squared prime factor,

then $\mu(mn) = 0$. Conversely, if both are square-free, the number of distinct prime factors of mnequals the sum of the numbers of distinct prime factors of $m$and n, so the sign pattern remains multiplicative.

Another fundamental property is the identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \qquad \text{(III)}$$

This identity establishes the Möbius function as the Dirichlet-convolution inverse of the constant-one function. Its role as an inverse makes it a fundamental instrument for reversing relationships involving sums over divisors and gives rise to the Möbius inversion formula (Haukkanen, 2023).

Dirichlet Convolution and the Algebraic Structure of Arithmetic Functions

Dirichlet convolution is a binary operation on two arithmetic functions $f$and g, defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right). \qquad \text{(IV)}$$

This operation connects the values of arithmetic functions with the divisor structure of an integer. Dirichlet convolution is commutative, associative, and distributive over addition. Under ordinary addition and Dirichlet convolution as multiplication, the set of arithmetic functions forms a commutative ring. The convolution identity is the function ε, defined by

$$\varepsilon(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \qquad \text{(V)}$$

Therefore, for every arithmetic function f,

$$f * \varepsilon = \varepsilon * f = f. \qquad \text{(VI)}$$

Every arithmetic function $f$satisfying $f(1) \neq 0$has a unique Dirichlet inverse, namely a function f^{-1}such that

$$f * f^{-1} = \varepsilon. \qquad \text{(VII)}$$

The inverse can be determined recursively. For $n = 1$,

$$f^{-1}(1) = \frac{1}{f(1)}, \qquad \text{(VIII)}$$

whereas for $n > 1$,

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d). \qquad \text{(IX)}$$

For the constant-one function $\mathbf{1}(n) = 1$, its Dirichlet inverse is the Möbius function. Hence,

$$1 * \mu = \mu * 1 = \varepsilon. \qquad \text{(X)}$$

This relationship makes it possible to interpret the Möbius function not merely as a function determined by prime factorization, but also as an inverse element in the algebra of arithmetic functions. This perspective is highly relevant to research on generalized Möbius functions because a generalized arithmetic function may be classified as a Möbius-type function when it acts as an inverse of another function under a particular convolution system.

Dirichlet convolution also preserves multiplicativity. If $f$and $g$are multiplicative functions, then $f * g$is also multiplicative. Furthermore, the Dirichlet inverse of a multiplicative function satisfying $f(1) \neq 0$remains multiplicative. This closure property allows the class of multiplicative functions to be analyzed as a group under Dirichlet convolution. Haukkanen (2023) demonstrated that convolution relations can also be analyzed locally at prime powers as Cauchy convolutions of the sequences $f(p^j)$and $g(p^j)$.

The Möbius Inversion Formula

The Möbius inversion formula is used to recover a function from the sum of its values over all divisors. Suppose the functions $F$and $f$satisfy

$$F(n) = \sum_{d|n} f(d). \qquad \text{(XI)}$$

This relationship can be written as

$$F = f * 1. \qquad \text{(XII)}$$

Convolving both sides with the Möbius function gives

$$F * \mu = f * 1 * \mu = f * \varepsilon = f. \qquad \text{(XIII)}$$

Therefore,

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right), \qquad \text{(XIV)}$$

or equivalently,

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d). \qquad \text{(XV)}$$

The formula demonstrates that the Möbius function acts as an inverse operator for divisor summation. In number theory, Möbius inversion is used to derive numerous arithmetic-function identities. For example, the identity function $N(n) = n$and Euler's totient function $\varphi(n)$satisfy

$$N = \varphi * 1. \qquad \text{(XVI)}$$

Through Möbius inversion, one obtains

$$\varphi = N * \mu, \qquad \text{(XVII)}$$

and consequently,

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}. \qquad \text{(XVIII)}$$

This relationship confirms that the Möbius function serves as an algebraic link among various arithmetic functions. A generalized Möbius function is expected to preserve a similar inversion capability, even when the underlying function, divisor system, or convolution operation is modified.

Generalizations of the Möbius Function

A generalization of the Möbius function can be constructed by extending its values at prime powers, introducing specific parameters, or modifying the convolution system. In general, an arithmetic function $\mu_\theta$may be regarded as a generalized Möbius function when it depends on a parameter θ, satisfies $\mu_\theta(1) = 1$, follows a consistent rule at prime powers, and possesses an inverse relationship with a particular arithmetic function.

One important generalization is Apostol's Möbius function of order k. This function is denoted by $\mu_k(n)$, with $\mu_1(n) = \mu(n)$. At prime powers, it can be defined as

$$\mu_k(p^\alpha) = \begin{cases} 1, & 0 \le \alpha < k, \\ -1, & \alpha = k, \\ 0, & \alpha > k. \end{cases} \qquad \text{(XIX)}$$

Its value at a general positive integer is then determined multiplicatively. This generalization changes the prime-exponent threshold at which the function becomes zero. In the classical Möbius function, the factor p^2already makes the value zero, whereas in the order-k Möbius function, zero occurs when an integer contains a prime power whose exponent exceeds k. Apostol used this function to investigate power-free integers and the behavior of partial sums of generalized Möbius functions. Subsequent studies have continued to refine the error terms in the asymptotic formulas associated with these partial sums (Banerjee et al., 2023).

The parameterization can be extended further. For instance, a generalized function may be defined at prime powers through a parameter sequence

$$\mu_\theta(p^\alpha) = c_{\theta,\alpha}, \qquad \text{(XX)}$$

and then extended multiplicatively by

$$\mu_\theta(n) = \prod_{i=1}^{r} c_{\theta,\alpha_i}, \; n = \prod_{i=1}^{r} p_i^{\alpha_i}. \qquad \text{(XXI)}$$

In this construction, the condition $c_{\theta,0} = 1$ensures that $\mu_\theta(1) = 1$. The values of $c_{\theta,\alpha}$determine the function's support, sign pattern, zero values, and convolution relationships. The classical Möbius function appears as a special case when

$$c_0 = 1, \; c_1 = -1, \; c_\alpha = 0 \text{ for } \alpha \ge 2. \qquad \text{(XXII)}$$

Accordingly, a meaningful generalization should satisfy a reduction principle, meaning that a particular parameter value restores the classical Möbius function.

In addition to modifying values at prime powers, generalization may be achieved by changing the convolution operation. Unitary, bi-unitary, infinitary, and incidence-based convolutions produce inverse functions different from the classical Möbius function. In each system, a generalized Möbius function is defined as the inverse of a zeta-type or constant-one function under the relevant convolution. Studies concerning extensions of the Möbius function demonstrate that changing the divisor concept and convolution operation produces different identities, inverses, and generalized totient functions (Sándor & Crstici, 2004).

Multiplicativity of Generalized Functions

Multiplicativity is one of the principal properties that must be examined in a generalized Möbius function. A function $f$is called multiplicative when $f(1) = 1$and

$$f(mn) = f(m)f(n) \qquad \text{(XXIII)}$$

for every $m, n \in \mathbb{N}$satisfying $\gcd(m, n) = 1$. If a generalized function is first defined at prime powers and its value at a general integer is then defined as the product of its local values, multiplicativity follows directly.

Suppose

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \qquad \text{(XXIV)}$$

and $\mu_\theta$is defined by

$$\mu_\theta(n) = \prod_{i=1}^{r} \mu_\theta\left(p_i^{\alpha_i}\right). \qquad \text{(XXV)}$$

For relatively prime integers $m$and n, their sets of prime factors are disjoint. Therefore,

$$\mu_\theta(mn) = \prod_{p^\alpha \| m} \mu_\theta\left(p^\alpha\right) \prod_{q^\beta \| n} \mu_\theta\left(q^\beta\right) = \mu_\theta(m)\mu_\theta(n). \qquad \text{(XXVI)}$$

This argument demonstrates that multiplicativity can be analyzed through local rules at prime powers. However, if the generalization parameter depends on relationships among prime factors or on global properties of n, multiplicativity is not automatically guaranteed. Therefore, consistency between the local and global definitions must constitute a central part of the mathematical analysis.

The study may also examine complete multiplicativity. A function $f$is completely multiplicative when $f(mn) = f(m)f(n)$for all positive integers $m$and n, without requiring them to be relatively prime. The classical Möbius function is not completely multiplicative because, for example,

$$\mu(p)^2 = 1, \qquad \text{(XXVII)}$$

whereas

$$\mu(p^2) = 0. \quad \text{(XXVIII)}$$

Generalizations that retain zero values at certain prime powers are generally not completely multiplicative either.

Identities and Inverses of Generalized Möbius Functions

The algebraic identities of a generalized function can be obtained by calculating its convolution with the constant-one function or another arithmetic function. Suppose a generalized function $\mu_\theta$has a corresponding function $z_\theta$satisfying

$$\mu_\theta * z_\theta = \varepsilon. \quad \text{(XXIX)}$$

Under this condition, $\mu_\theta$is the Dirichlet inverse of z_θ. If $z_\theta = \mathbf{1}$, the uniqueness of the Dirichlet inverse requires that $\mu_\theta = \mu$. Therefore, a function genuinely different from the classical Möbius function must either be the inverse of a different base function or operate under a generalized convolution.

For multiplicative functions, the inverse relationship can be verified at each prime power. If

$$\sum_{j=0}^{a} \mu_\theta\left(p^j\right) z_\theta\left(p^{a-j}\right) = \begin{cases} 1, & a = 0, \\ 0, & a \geq 1, \end{cases} \quad \text{(XXX)}$$

then globally,

$$\mu_\theta * z_\theta = \varepsilon. \quad \text{(XXXI)}$$

This local approach simplifies the proof because the convolution problem over all positive integers is reduced to a Cauchy-convolution problem involving sequences of prime-power values. The correspondence between Dirichlet convolution at $p^n$and Cauchy convolution allows inverses and identities of arithmetic functions to be determined through local generating functions (Haukkanen, 2023).

The local generating function, or Bell series, of a multiplicative function $f$at a prime $p$is defined by

$$\mathcal{B}_p(f; x) = \sum_{\alpha=0}^{\infty} f\left(p^\alpha\right) x^\alpha. \quad \text{(XXXII)}$$

If $h = f * g$, then (XXXIII)

$$\mathcal{B}_p(h; x) = \mathcal{B}_p(f; x)\mathcal{B}_p(g; x). \quad \text{(XXXIV)}$$

Consequently, if $g = f^{-1}$,

$$\mathcal{B}_p(g; x) = \frac{1}{\mathcal{B}_p(f; x)}. \quad \text{(XXXV)}$$

This framework can be applied to determine the inverse partner of a generalized Möbius function. Moreover, the form of the generating function can indicate whether the coefficients of the generalized function are finite, periodic, or governed by a particular recurrence relation.

Relationships with Other Arithmetic Functions

The Möbius function is closely related to several other arithmetic functions. For example, the indicator function of square-free integers can be written as

$$\mu^2(n) = \begin{cases} 1, & n \text{ is square-free,} \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(XXXVI)}$$

The Liouville function, defined by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$denotes the number of prime factors counted with multiplicity, is also related to the Möbius function through prime-factorization structure. The principal difference is that the Liouville function does not vanish on integers containing squared prime factors. Therefore, a generalized Möbius function can be analyzed by comparing its support and sign pattern with those of the Liouville function.

Another important relationship involves Euler's totient function. The identity

$$\varphi = N * \mu \qquad \text{(XXXVII)}$$

shows that the totient function is the Dirichlet convolution of the identity function $N(n) = n$and the Möbius function. More generally, Jordan's totient function of order $k$satisfies

$$J_k = N_k * \mu,\ N_k(n) = n^k. \qquad \text{(XXXVIII)}$$

Its explicit form is

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right). \qquad \text{(XXXIX)}$$

When the Möbius function is replaced by a generalized function μ_θ, a generalized totient function may be defined by

$$J_{k,\theta} = N_k * \mu_\theta. \qquad \text{(XL)}$$

The properties of this new function depend on the local values $\mu_\theta(p^\alpha)$. An analysis of this relationship may generate new divisor identities and extend the combinatorial interpretation of totient functions.

The connection with Dirichlet series is also important for explaining the analytic properties of the Möbius function. For Re $(s) > 1$, the Dirichlet series of the classical Möbius function satisfies

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}. \qquad \text{(XLI)}$$

This relationship follows from the identity $\mu * \mathbf{1} = \varepsilon$, since Dirichlet convolution of coefficients corresponds to multiplication of Dirichlet series. For a generalized multiplicative Möbius function, its Dirichlet series can be expressed as an Euler product:

$$\sum_{n=1}^{\infty} \frac{\mu_\theta(n)}{n^s} = \prod_p \left(\sum_{\alpha=0}^{\infty} \frac{\mu_\theta(p^\alpha)}{p^{\alpha s}}\right), \qquad \text{(XLII)}$$

provided that the series converges absolutely. The local factors in the Euler product can be used to identify relationships between the generalized function and particular zeta or L-functions.

The Möbius Function in Incidence Algebra

The concept of the Möbius function can be extended from positive integers to partially ordered sets. In incidence algebra, the zeta function is defined on intervals of a partially ordered set, while the Möbius function is its convolution inverse. This framework demonstrates that Möbius inversion in number theory is a special case of a broader inversion principle.

On the set of positive integers ordered by divisibility, $a \preceq b$when $a \mid b$, the convolution in incidence algebra corresponds to Dirichlet convolution. The Möbius function of the interval $[a, b]$can be written as

$$\mu(a, b) = \mu\left(\frac{b}{a}\right) \qquad \text{(XLIII)}$$

whenever $a \mid b$. This result shows that the classical Möbius function arises from the divisibility structure of positive integers rather than merely from a sign rule based on prime factorization. Rota (1964) developed Möbius theory on partially ordered sets as a general framework for combinatorial inversion, while more recent developments have extended incidence algebra and Möbius inversion to more abstract decomposition structures (Gálvez-Carrillo et al., 2018).

The incidence-algebra perspective suggests three principal approaches to generalizing the Möbius function. First, the values at prime powers may be modified while retaining Dirichlet convolution. Second, the domain of positive integers may be retained while the concepts of divisors or convolution are changed. Third, the divisibility relation may be replaced with another partial-order relation. Each approach produces different identities and inversion formulas, but all remain centered on the principle that the Möbius function is the inverse of a zeta function in a convolution algebra.

Conceptual Framework of the Study

Based on the theoretical discussion, a generalized Möbius-type arithmetic function may be formulated as a parameterized function.

$$\boldsymbol{\mu_\theta}: \mathbb{N} \to \mathbb{C}, \qquad \text{(XLIV)}$$

defined through local values at prime powers. The form or parameter of the generalization constitutes the independent variable and determines the rule

$$\mu_\theta(p^\alpha) = c_{\theta,\alpha}. \qquad \text{(XLV)}$$

The algebraic properties serving as dependent variables include multiplicativity, Dirichlet convolution, identity elements, the existence of inverses, inversion formulas, and relationships with other arithmetic functions. Meanwhile, the domain of positive integers, the uniqueness of prime factorization, and the definition of the classical Möbius function serve as the mathematical foundation and control conditions.

The analysis begins by confirming that $\mu_\theta(1) = 1$and that the classical Möbius function is recovered for a particular parameter value. Multiplicativity is then examined through prime factorization. The convolution $\mu_\theta * f$is calculated locally at $p^\alpha$to identify the corresponding inverse function and the resulting identities. The generalized function is subsequently compared with the classical Möbius function, the Liouville function, power-free indicator functions, totient functions, and its associated Dirichlet series. A generalization may be considered algebraically robust when it satisfies definitional consistency, the reduction principle, multiplicative closure, the existence of an inverse, and the ability to generate an applicable inversion formula for a particular class of arithmetic functions.

3. RESEARCH METHODS

This study employs a qualitative approach with a literature review design and mathematical-theoretical analysis. This approach was selected because the research does not involve empirical measurements of respondents but focuses on conceptual development, definition formulation, proposition construction, and the proof of algebraic properties of Möbius-type arithmetic functions. The primary objects of the study are arithmetic functions obtained through generalizations of the classical Möbius function, particularly those related to prime factorization structures, multiplicativity, Dirichlet convolution, identity elements, Dirichlet inverses, and relationships with other arithmetic functions. The analysis is conducted over the domain of positive integers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \text{(XLVI)}$$

with codomains consisting of real numbers, complex numbers, or a commutative ring with an identity element. This algebraic structure allows the generalized functions to be examined as elements of an algebra of arithmetic functions under addition and convolution operations.

The research data consist of secondary data obtained from scientific articles, number theory books, conference proceedings, dissertations, theses, and mathematical publications discussing the Möbius function, Möbius inversion, multiplicative functions, Dirichlet convolution, and generalizations of arithmetic functions. The literature is selected purposively

according to its relevance to the research focus. The inclusion criteria cover sources containing formal definitions, theorems, propositions, lemmas, proofs, or identities directly related to the Möbius function and operations on arithmetic functions. Sources discussing Möbius concepts exclusively in geometry, complex transformations, or topology are excluded because they fall outside the scope of number theory. Literature searches are conducted using keywords such as *Möbius function*, *generalized Möbius function*, *arithmetic function*, *multiplicative function*, *Dirichlet convolution*, *Dirichlet inverse*, and *Möbius inversion*. Studies on generalized convolution are also examined because modifications to convolution operations may produce Möbius-type inverse functions that differ from the classical Möbius function.

The units of analysis include definitions, generalization parameters, formulas, identities, theorems, propositions, and mathematical proof procedures. This study does not involve research subjects or informants because it does not use interviews, field observations, or experiments involving human participants. Although the terms independent and dependent variables are used to indicate the direction of analysis, these variables are not treated statistically. The independent variable $X$is represented by the form or parameters of the generalized Möbius-type arithmetic function, whereas the dependent variable Yconsists of the algebraic properties produced by the generalization. The domain of positive integers, prime factorization structure, and the definition of the classical Möbius function are treated as controlled mathematical conditions.

The classical Möbius function is used as the starting point and is defined as follows:

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = p_1 p_2 \cdots p_r, \\ 0, & p^2 \mid n \text{ for some prime } p, \end{cases} \qquad \text{(XLVII)}$$

where $p_1, p_2, \ldots, p_r$are distinct prime numbers. Based on this definition, the study constructs a family of generalized arithmetic functions expressed as

$$\mu_\theta : \mathbb{N} \to R, \qquad \text{(XLVIII)}$$

where θdenotes one or more generalization parameters and $R$is a commutative ring with an identity element. The initial form of the generalization can be determined through its values at prime powers:

$$\mu_\theta(p^a) = \Phi_\theta(p, a). \qquad \text{(XLIX)}$$

The function is subsequently extended to every positive integer

$$n = \prod_{i=1}^{r} p_i^{a_i} \qquad \text{(L)}$$

through the expression

$$\mu_\theta(n) = \prod_{i=1}^{r} \Phi_\theta\,(p_i, a_i), \qquad \text{(LI)}$$

provided that the constructed function satisfies multiplicativity. This formulation is not immediately accepted as a final result but is tested for consistency with the classical Möbius function. One possible consistency condition is the existence of a particular parameter $\theta_0$such that

$$\mu_{\theta_0}(n) = \mu(n) \qquad \text{(LII)}$$

for every $n \in \mathbb{N}$.

Data are collected through documentation and mathematical note-taking techniques. Each relevant source is recorded in an analytical matrix containing the source identity, function definition, domain and codomain, form of generalization, algebraic operation, assumptions, theorem results, and proof patterns. The extracted results are then classified into five main categories: multiplicativity, Dirichlet convolution, identity elements, inverse functions, and relationships with other arithmetic functions. This classification is used to compare similarities and differences among various generalizations and to identify the minimum conditions required for a function to retain Möbius-type properties.

The multiplicativity analysis examines whether the generalized function satisfies

$$\mu_\theta(mn) = \mu_\theta(m)\mu_\theta(n), \text{ gcd } (m, n) = 1. \qquad \text{(LIII)}$$

When this equation holds for every pair of relatively prime integers $m$and n, the function is classified as multiplicative. The analysis focuses on the values of the function at prime powers because a multiplicative arithmetic function is determined by its values at p^a. Accordingly, the proof is conducted by considering the prime factorizations

$$m = \prod_{i=1}^{r} p_i^{\alpha_i},\ n = \prod\prod_{j=1}^{s} q_j^{\beta_j}, \qquad \text{(LIV)}$$

where $p_i \neq q_j$, and then comparing the value of $\mu_\theta(mn)$with $\mu_\theta(m)\mu_\theta(n)$.

The convolution analysis uses the Dirichlet convolution operation. For two arithmetic functions $f$and g, their Dirichlet convolution is defined by

$$(f * g)(n) = \sum_{d|n} f\,(d) g\left(\frac{n}{d}\right). \qquad \text{(LV)}$$

The identity element under Dirichlet convolution is represented by

$$\varepsilon(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases} \qquad \text{(LVI)}$$

so that

$$f * \varepsilon = \varepsilon * f = f. \qquad \text{(LVII)}$$

The classical Möbius function is the Dirichlet inverse of the constant function $1(n) = 1$. Therefore.

$$\mu * 1 = 1 * \mu = \varepsilon. \quad \text{(LVIII)}$$

This relationship serves as a benchmark for determining whether a generalized function can act as the inverse of a particular arithmetic function. In general, when there exists a function $h_\theta$satisfying

$$\mu_\theta * h_\theta = \varepsilon, \quad \text{(LIX)}$$

then $\mu_\theta$is the Dirichlet inverse of h_θ, written as

$$\mu_\theta = h_\theta^{-1}. \quad \text{(LX)}$$

The existence of the inverse requires $h_\theta(1)$to be an invertible element. For complex-valued functions, this condition becomes

$$h_\theta(1) \neq 0. \quad \text{(LXI)}$$

The inverse values can then be determined recursively through

$$h_\theta^{-1}(1) = \frac{1}{h_\theta(1)} \quad \text{(LXII)}$$

and, for $n > 1$,

$$h_\theta^{-1}(n) = -\frac{1}{h_\theta(1)} \sum_{\substack{d|n \\ d<n}} h_\theta\left(\frac{n}{d}\right) h_\theta^{-1}(d). \quad \text{(LXIII)}$$

Möbius inversion analysis is conducted by examining the relationship between two arithmetic functions. In the classical form, when

$$G(n) = \sum_{d|n} F(d), \quad \text{(LXIV)}$$

or equivalently,

$$G = 1 * F, \quad \text{(LXV)}$$

then

$$F = \mu * G. \quad \text{(LXVI)}$$

Consequently,

$$F(n) = \sum_{d|n} \mu(d) G\left(\frac{n}{d}\right). \quad \text{(LXVII)}$$

For generalized functions, this relationship is developed into

$$G = h_\theta * F. \quad \text{(LXVIII)}$$

When $\boldsymbol{h_\theta}$possesses a Dirichlet inverse, it follows that

$$F = h_\theta^{-1} * G = \mu_\theta * G. \quad \text{(LXIX)}$$

Therefore,

$$F(n) = \sum_{d|n} \mu_\theta(d) G\left(\frac{n}{d}\right). \quad \text{(LXX)}$$

This framework is used to determine whether the constructed function produces a valid Möbius inversion principle rather than merely a symbolic modification of the classical formula.

Generalized Möbius inversion is also considered in relation to incidence algebras and other algebraic structures that provide a conceptual basis for extending the Möbius function beyond its classical form.

The relationship between the generalized function and other arithmetic functions is analyzed through convolution identities. Several classical identities are used as points of comparison, including

$\varphi = \mu * \mathrm{id},$ (LXXI)

where

$\mathrm{id}\,(n) = n,$ (LXXII)

and

$\tau = 1 * 1,$ (LXXIII)

where $\boldsymbol{\tau(n)}$denotes the number of positive divisors of $\boldsymbol{n}$. Based on these patterns, the study investigates possible identities of the form

$F_\theta = \mu_\theta * g$ (LXXIV)

or

$\mu_\theta * F_\theta = G_\theta,$ (LXXV)

for suitable arithmetic functions $\boldsymbol{g}$, $\boldsymbol{F_\theta}$, and $\boldsymbol{G_\theta}$.

The analysis also considers Dirichlet series representations. If

$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$ (LXXVI)

then, in a region of absolute convergence,

$D_{f*g}(s) = D_f(s) D_g(s).$ (LXXVII)

Therefore, when

$\mu_\theta = h_\theta^{-1},$ (LXXVIII)

the following relationship is obtained:

$D_{\mu_\theta}(s) = \frac{1}{D_{h_\theta}(s)},$ (LXXIX)

provided that $D_{h_\theta}(s) \neq 0$. This representation is used as an additional verification of the convolution identities derived in the study.

The data analysis is conducted through four stages. The first stage is conceptual reduction, which involves selecting definitions and mathematical results relevant to the Möbius function and excluding discussions unrelated to number theory. The second stage is structural comparison, which compares classical and generalized functions based on their values at 1, values at prime powers, functional support, multiplicative properties, and convolution inverses.

The third stage is mathematical deduction, which derives new lemmas, propositions, and theorems from the formulated definitions. The fourth stage is proof verification, which examines each result through direct substitution, prime factorization, the associative and commutative properties of convolution, and tests involving basic cases such as $n = 1$, $n = p$, $n = p^2$, $n = pq$, and $n = p^a q^b$.

The validity of the research findings is maintained through definitional consistency, cross-checking among sources, and formal verification of every step of the mathematical proofs. A proposition is considered valid when all assumptions are stated explicitly, each algebraic transformation follows an established definition or theorem, and the conclusion applies throughout the specified domain. In addition, the generalized function must satisfy a reduction test, meaning that it returns to the classical Möbius function for a particular parameter value. To minimize errors, the results are also tested using small positive integers and compared with established convolution identities. Thus, the validity of this study is not determined by statistical significance but by logical accuracy, symbolic consistency, adequacy of assumptions, and the reproducibility of mathematical proofs.

4. RESULTS AND DISCUSSION

Construction of a Parameterized Möbius-Type Arithmetic Function

Based on the mathematical-theoretical analysis, this study formulates a family of Möbius-type arithmetic functions indexed by a parameter $t \in \mathbb{C}$. The function is denoted by $\mu_t: \mathbb{N} \to \mathbb{C}$and defined locally at prime powers as

$\mu_t(1) = 1,$

$$\mu_t(p^a) = \begin{cases} -t, & a = 1, \\ 0, & a \geq 2, \end{cases} \qquad \text{(LXXX)}$$

for every prime number p. For an integer with prime factorization

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \qquad \text{(LXXXI)}$$

the function is extended multiplicatively by

$$\mu_t(n) = \prod_{i=1}^{r} \mu_t\left(p_i^{a_i}\right). \qquad \text{(LXXXI)}$$

Consequently, the explicit form of the generalized function is

$$\mu_t(n) = \begin{cases} 1, & n = 1, \\ (-t)^{\omega(n)}, & n \text{ is square-free,} \\ 0, & n \text{ is not square-free,} \end{cases} \qquad \text{(LXXXII)}$$

where $\omega(n)$denotes the number of distinct prime factors of n. This formulation preserves the support of the classical Möbius function on square-free integers while allowing its sign and magnitude to vary according to the parameter t. The construction is consistent with the conceptual framework and prime-power analysis described in the research method.

The reduction principle is fulfilled when $t = 1$, because

$\mu_1(n) = \mu(n).$ (LXXXIII)

Therefore, the classical Möbius function is obtained as a special member of the generalized family. Another important case occurs when $t = -1$, yielding

$\mu_{-1}(n) = \mu^2(n),$ (LXXXIV)

which is the characteristic function of square-free integers. When $t = 0$, the function reduces to the convolution identity ε, because $\mu_0(1) = 1$and $\mu_0(n) = 0$for $n > 1$. These special cases demonstrate that variation of the parameter does not merely change numerical values but produces connections among the Möbius function, the square-free indicator function, and the identity element of the Dirichlet-convolution algebra.

Table 1. Values of the Generalized Möbius-Type Function.

n	Prime factorization	$\mu_t(n)$
1	1	1
2	2	$-t$
3	3	$-t$
4	2^2	0
5	5	$-t$
6	$2 \cdot 3$	t^2
8	2^3	0
10	$2 \cdot 5$	t^2
12	$2^2 \cdot 3$	0
30	$2 \cdot 3 \cdot 5$	$-t^3$

The table confirms that the function vanishes whenever an integer contains a squared prime factor. For square-free integers, the exponent of $t$equals the number of distinct prime factors, while the sign is determined by the parity of that number. This result shows that the proposed generalization retains the structural dependence on prime factorization that characterizes the classical Möbius function.

Multiplicativity of the Generalized Function

The first algebraic property examined is multiplicativity. Let $m$and $n$be positive integers satisfying gcd $(m, n) = 1$. Their prime factorizations may be written as.

$$m = \prod_{i=1}^{r} p_i^{\alpha_i},\ n = \prod_{j=1}^{s} q_j^{\beta_j}, \qquad \text{(LXXXV)}$$

where the prime sets $\{p_1, \ldots, p_r\}$and $\{q_1, \ldots, q_s\}$are disjoint. It follows that

$$mn = \prod_{i=1}^{r} p_i^{\alpha_i} \prod_{j=1}^{s} q_j^{\beta_j}. \qquad \text{(LXXXVI)}$$

Using the local definition at prime powers,

$$\begin{aligned} \mu_t(mn) &= \prod_{i=1}^{r} \mu_t\left(p_i^{\alpha_i}\right) \prod_{j=1}^{s} \mu_t\left(q_j^{\beta_j}\right) \\ &= \mu_t(m)\mu_t(n). \end{aligned} \qquad \text{(LXXXVII)}$$

Thus, $\mu_t$is multiplicative for every $t \in \mathbb{C}$. This finding confirms that defining a generalized arithmetic function through independent local values at prime powers is sufficient to preserve multiplicativity, provided that the global value is constructed as the product of the corresponding local values.

However, $\mu_t$is generally not completely multiplicative. For a prime p,

$$\mu_t(p)^2 = t^2, \qquad \text{(LXXXVIII)}$$

whereas

$$\mu_t(p^2) = 0. \qquad \text{(LXXXIX)}$$

Therefore,

$$\mu_t(p^2) \neq \mu_t(p)\mu_t(p) \qquad \text{(XC)}$$

whenever $t \neq 0$. Complete multiplicativity occurs only in the degenerate case $t = 0$. This distinction is important because it shows that the generalized function preserves the ordinary multiplicative property of the classical Möbius function but not complete multiplicativity.

The result is consistent with generalized Möbius-function studies in which local rules at prime powers determine global multiplicative behavior. Generalizations introduced through order parameters or modified prime-power values commonly preserve multiplicativity, although their support and convolution partners may differ from those of the classical function. The literature also indicates that generalized Möbius functions can be used to characterize completely multiplicative functions through specific convolution equations rather than by requiring the Möbius-type function itself to be completely multiplicative (Laohakosol et al., 2012).

Dirichlet Inverse and Convolution Identity

The next result identifies the arithmetic function whose Dirichlet inverse is μ_t. Define

$$z_t(n) = t^{\Omega(n)}, \qquad \text{(XCI)}$$

where $\Omega(n)$represents the number of prime factors of n, counted with multiplicity. Since

$$z_t(p^a) = t^a, \qquad \text{(XCII)}$$

the Bell series of $z_t$at a prime $p$is

$$B_p(z_t; x) = \sum_{a=0}^{\infty} t^a\, x^a = \frac{1}{1-tx}. \qquad \text{(XCIII)}$$

Meanwhile, the Bell series of $\mu_t$is

$$B_p(\mu_t; x) = 1 - tx, \qquad \text{(XCIV)}$$

because $\mu_t(p) = -t$and $\mu_t(p^a) = 0$for $a \geq 2$. Therefore,

$$B_p(\mu_t; x)B_p(z_t; x) = (1 - tx)\frac{1}{1-tx} = 1. \qquad \text{(XCV)}$$

The multiplication rule for Bell series consequently gives

$$\mu_t * z_t = \varepsilon. \qquad \text{(XCVI)}$$

Hence,

$$\boxed{\mu_t = z_t^{-1}} \qquad \text{(XCVII)}$$

under Dirichlet convolution. The same result can be verified directly at prime powers:

$$(\mu_t * z_t)(p^a) = \sum_{j=0}^{a} \mu_t\left(p^j\right) z_t\left(p^{a-j}\right). \qquad \text{(XCVIII)}$$

For $a = 0$,

$$(\mu_t * z_t)(1) = 1. \qquad \text{(XCIX)}$$

For $a \geq 1$, only the terms $j = 0$and $j = 1$are nonzero, so

$$\begin{aligned}(\mu_t * z_t)(p^a) &= \mu_t(1)z_t(p^a) + \mu_t(p)z_t(p^{a-1}) \\ &= t^a - t \cdot t^{a-1} \\ &= 0.\end{aligned} \qquad \text{(C)}$$

Since both functions are multiplicative, equality at every prime power implies equality for all positive integers.

This finding provides a clear algebraic interpretation of the parameter t. The generalized function is not, except when $t = 1$, the inverse of the constant-one function. Instead, it is the Dirichlet inverse of the completely multiplicative function $z_t(n) = t^{\Omega(n)}$. When $t = 1$,

$$z_1(n) = 1, \qquad \text{(CI)}$$

and the classical identity

$$\mu * 1 = \varepsilon \qquad \text{(CII)}$$

is recovered. When $t = -1$,

$$z_{-1}(n) = (-1)^{\Omega(n)} = \lambda(n), \qquad \text{(CIII)}$$

where $\lambda$is the Liouville function. Consequently,

$$\mu_{-1} * \lambda = \mu^2 * \lambda = \varepsilon. \qquad \text{(CIV)}$$

This relation explains algebraically why the square-free indicator function is the Dirichlet inverse of the Liouville function.

The finding supports Haukkanen's conclusion that convolution equations and inverse arithmetic functions can be analyzed locally through sequences of values at prime powers. An arithmetic function possesses a unique Dirichlet inverse when its value at 1is nonzero; the corresponding inverse may be obtained recursively or through local convolution analysis.

Generalized Möbius Inversion Formula

Because $\mu_t * z_t = \varepsilon$, a generalized inversion formula can be established. Suppose two arithmetic functions $F$and Gsatisfy

$$G = z_t * F. \qquad \text{(CV)}$$

In explicit form,

$$G(n) = \sum_{d|n} t^{\Omega(d)} F\left(\frac{n}{d}\right). \qquad \text{(CVI)}$$

Convolving both sides with $\mu_t$gives

$$\mu_t * G = \mu_t * z_t * F = \varepsilon * F = F. \qquad \text{(CVII)}$$

Therefore,

$$\boxed{F(n) = \sum_{d|n} \mu_t\,(d) G\left(\frac{n}{d}\right)} \qquad \text{(CVIII)}$$

or equivalently,

$$\boxed{F(n) = \sum_{d|n} \mu_t\left(\frac{n}{d}\right) G(d).} \qquad \text{(CIX)}$$

When $t = 1$, the formula becomes

$$G(n) = \sum_{d|n} F\,(d) \iff F(n) = \sum_{d|n} \mu\,(d) G\left(\frac{n}{d}\right), \qquad \text{(CX)}$$

which is the classical Möbius inversion formula. Thus, the generalized formula satisfies the reduction principle and provides a valid inverse transformation for weighted divisor sums involving $t^{\Omega(d)}$.

This result shows that a generalized Möbius function should not be evaluated merely according to whether it resembles the classical sign pattern. Its essential algebraic characteristic is its ability to invert a corresponding divisor-summation operator. Earlier research also demonstrates that Möbius inversion can be extended when the generalized functions and convolution operations satisfy appropriate algebraic conditions, including associativity, identity, and inverse existence (He et al., 2006).

Relationships with Other Arithmetic Functions

The generalized function produces a parameterized form of Euler's totient function. Define

$$\varphi_t = id * \mu_t, \qquad \text{(CXI)}$$

where $id(n) = n$. Then

$$\varphi_t(n) = \sum_{d|n} \mu_t(d)\frac{n}{d}. \qquad \text{(CXII)}$$

Because both $id$and $\mu_t$are multiplicative, $\varphi_t$is also multiplicative. At a prime power p^a, $a \geq 1$,

$$\begin{aligned}\varphi_t(p^a) &= \mu_t(1)p^a + \mu_t(p)p^{a-1} \\ &= p^a - tp^{a-1} \\ &= p^a\left(1 - \frac{t}{p}\right).\end{aligned} \qquad \text{(CXIII)}$$

Consequently, for

$$n = \prod_{i=1}^{r} p_i^{a_i}, \qquad \text{(CXIV)}$$

one obtains

$$\boxed{\varphi_t(n) = n\prod_{p|n}\left(1 - \frac{t}{p}\right).} \qquad \text{(CXV)}$$

When $t = 1$, this formula reduces to Euler's totient function:

$$\varphi_1(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) = \varphi(n). \qquad \text{(CXVI)}$$

When $t = -1$,

$$\varphi_{-1}(n) = n\prod_{p|n}\left(1 + \frac{1}{p}\right), \qquad \text{(CXVII)}$$

which is the Dedekind psi function. Therefore, the generalized family provides a direct bridge between Euler's totient function and the Dedekind psi function.

The construction can be extended to a generalized Jordan-type totient function. Let $id_k(n) = n^k$and define

$$J_{k,t} = id_k * \mu_t. \qquad \text{(CXVIII)}$$

At a prime power,

$$J_{k,t}(p^a) = p^{ak} - tp^{(a-1)k}, \qquad \text{(CXIX)}$$

so globally,

$$\boxed{J_{k,t}(n) = n^k\prod_{p|n}\left(1 - \frac{t}{p^k}\right).} \qquad \text{(CXX)}$$

For $t = 1$, the classical Jordan totient function is recovered. This result demonstrates that changing one local coefficient in a Möbius-type function generates a broader family of totient functions while preserving multiplicativity and product representations.

The ability of generalized Möbius functions to generate new arithmetic identities is consistent with previous work on generalized Ramanujan sums. Replacing the classical Möbius function with another Möbius-type inverse has been shown to produce modified representations, orthogonality properties, inversion formulas, and arithmetic applications (Laohakosol et al., 2006).

Dirichlet-Series Representation

The Dirichlet series of $\mu_t$can be determined through its Euler product. In a region of absolute convergence,

$$\sum_{n=1}^{\infty} \frac{\mu_t(n)}{n^s} = \prod_p \left(\sum_{a=0}^{\infty} \frac{\mu_t(p^a)}{p^{as}} \right). \qquad \text{(CXXI)}$$

Because the only nonzero local values occur at $a = 0$and $a = 1$,

$$\boxed{\sum_{n=1}^{\infty} \frac{\mu_t(n)}{n^s} = \prod_p \left(1 - \frac{t}{p^s}\right).} \qquad \text{(CXXII)}$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{z_t(n)}{n^s} = \prod_p \left(1 - \frac{t}{p^s}\right)^{-1}. \qquad \text{(CXXIII)}$$

Thus,

$$D_{\mu_t}(s) D_{z_t}(s) = 1, \qquad \text{(CXXIV}$$

which is the Dirichlet-series equivalent of

$$\mu_t * z_t = \varepsilon. \qquad \text{CXXV)}$$

For $t = 1$,

$$D_{\mu_1}(s) = \prod_p (1 - p^{-s}) = \frac{1}{\zeta(s)}. \qquad \text{(CXXVI)}$$

The generalized Euler product cannot, for arbitrary t, always be expressed as a simple integer power of the Riemann zeta function. Nevertheless, it remains a valid local-factor representation and directly reflects the values of $\mu_t$at prime powers.

Overall Discussion

The results demonstrate that the algebraic behavior of a generalized Möbius-type function is determined primarily by its local coefficients at prime powers. The condition $\mu_t(1) = 1$guarantees the normalization required for multiplicative arithmetic functions, while the values $\mu_t(p) = -t$and $\mu_t(p^a) = 0$for $a \geq 2$determine its support, sign pattern, inverse

partner, and Dirichlet series. This confirms that prime-power analysis is the most efficient method for investigating generalized arithmetic functions.

The proposed generalization preserves four essential characteristics of the classical Möbius function. First, it is multiplicative. Second, it reduces to the classical function at $t = 1$. Third, it possesses a unique inverse relationship under Dirichlet convolution. Fourth, it generates a valid inversion formula and parameterized totient identities. These properties indicate that $\mu_t$is not merely a symbolic modification but an algebraically consistent Möbius-type arithmetic function.

The results further reveal that a nonclassical Möbius-type function cannot remain the inverse of the constant-one function under ordinary Dirichlet convolution. The uniqueness of the Dirichlet inverse implies that any function satisfying $f * 1 = \varepsilon$must equal the classical Möbius function. Therefore, a meaningful generalization must change either the inverse partner, as in the present construction, or the convolution system. This conclusion explains why generalized Möbius theories frequently employ weighted base functions, unitary divisors, modified convolutions, or broader incidence algebras. Research on unitary convolution similarly shows that changing the divisor relation produces different Möbius inverses and inversion formulas.

From the incidence-algebra perspective, these findings retain the fundamental principle that a Möbius function is the convolution inverse of a zeta-type function. In the divisibility poset, the classical zeta function corresponds to the constant-one arithmetic function, whereas the present construction replaces it with the weighted function $z_t(n) = t^{\Omega(n)}$. The result remains consistent with the broader interpretation of Möbius inversion as an inverse operation in a convolution algebra, originally formalized in the theory of partially ordered sets (Rota, 1964).

Overall, the generalization parameter $t$acts as an algebraic controller connecting several familiar arithmetic functions. The value $t = 0$produces the convolution identity, $t = 1$produces the classical Möbius function and Euler's totient function, and $t = -1$produces the square-free indicator and Dedekind psi function. Therefore, the principal contribution of this study lies in providing an integrated framework in which changes in a Möbius-type parameter can be traced systematically through multiplicativity, convolution inverses, inversion formulas, Euler products, and relationships with other arithmetic functions.

5. CONCLUSIONS AND RECOMMENDATIONS

The conclusion should be written concisely, addressing the research objectives or problems by presenting the research findings or hypothesis-testing results without repeating the discussion. The conclusion should be stated critically, logically, and honestly based on the available research evidence, with due caution when attempting to generalize the findings. The conclusion and recommendations should be presented in paragraph form, without numbering or bullet points. This section may also include suggestions or recommendations for actions based on the research conclusions. Likewise, authors are strongly encouraged to discuss the limitations of the study and provide recommendations for future research.

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